3 forms of **convexity** in **graphs & networks**

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joint work with

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COSTNET ´17
definitions of convexity

**convex/non-convex** real functions, sets in \( \mathbb{R}^2 \) & subgraphs

disconnected \( \supseteq \) connected \( \supseteq \) induced \( \supseteq \) isometric \( \supseteq \) convex subgraphs

(sna) \( k \)-clubs & \( k \)-clans are **convex** \( k \)-cliques

(def) **subset** \( S \) is convex if it induces convex **subgraph**

(def) convex **hull** \( \mathcal{H}(S) \) is smallest convex subset including \( S \)
expansion of convex subsets

grow subset $S$ by one node & expand $S$ to convex hull $\mathcal{H}(S)$

- $S = \{\text{random node } i\}$
- until $S$ contains $n$ nodes:
  1. select $i \notin S$ by random edge
  2. expand $S = \mathcal{H}(S \cup \{i\})$

$S$ quantifies (locally) tree-like/clique-like structure of graphs
convex expansion in graphs

\[ s(t) = \text{average fraction of nodes in } S \text{ after } t \text{ expansion steps} \]

\[ s(t) = \frac{(t + 1)}{n} \text{ in convex } \& \ s(t) \gg \frac{(t + 1)}{n} \text{ in non-convex graphs} \]

\[ s(t) \text{ quantifies (locally) tree-like/clique-like structure of graphs} \]
random graphs fail to reproduce convexity in empirical networks
random graphs convex for \(< \mathcal{O}(\ln n)\) & non-convex for \(> \mathcal{O}(\ln^2 n)\)
core-periphery networks have convex periphery & non-convex c-core
global measure \( c \)-convexity

\[
X_c = 1 - \sum_{t=1}^{n-1} \sqrt{\max(\Delta s(t) - 1/n, 0)}
\]

\( X_c \geq X_c^{RW} \geq X_c^{ER} \)

\( X_c \) highlights tree-like/clique-like networks (cliques connected tree-like)

\[
\begin{array}{ccccccc}
& X_1 & X_1^{RW} & X_1^{ER} & X_{1.1} & X_{1.1}^{RW} & X_{1.1}^{ER} \\
\hline
\text{Western US power grid}^{*} & 0.95 & 0.32 & 0.24 & 0.91 & 0.10 & 0.01 \\
\text{European highways}^{*} & 0.66 & 0.23 & 0.27 & 0.44 & -0.02 & 0.06 \\
\text{Networks coauthorships} & 0.91 & 0.09 & 0.06 & 0.83 & -0.05 & -0.09 \\
\text{Oregon Internet map} & 0.68 & 0.36 & 0.06 & 0.53 & 0.20 & -0.09 \\
\text{Caenorhabditis elegans} & 0.57 & 0.54 & 0.07 & 0.43 & 0.40 & -0.13 \\
\text{US airports connections} & 0.43 & 0.24 & 0.00 & 0.30 & 0.16 & -0.07 \\
\text{Scientometrics citations} & 0.24 & 0.16 & 0.02 & 0.04 & 0.00 & -0.13 \\
\text{US election weblogs} & 0.17 & 0.12 & 0.00 & 0.06 & 0.04 & -0.08 \\
\text{Little Rock food web} & 0.03 & 0.03 & 0.02 & -0.06 & -0.02 & -0.02 \\
\end{array}
\]

\( X_c \) measures global & regional (periphery) convexity in networks
local measure of convexity

\[ L_c = 1 + \max\{ t \mid s(t) < (t + c + 1)/n \} \quad L_1 \leq L_1^{\text{ER}} \approx \ln n / \ln \langle k \rangle \]

\( L_c \) highlights locally **tree-like/clique-like** networks & random graphs

<table>
<thead>
<tr>
<th>Network</th>
<th>( L_t )</th>
<th>( L_t^{\text{ER}} )</th>
<th>( L_1 )</th>
<th>( L_1^{\text{ER}} )</th>
<th>( \ln n / \ln \langle k \rangle )</th>
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<tbody>
<tr>
<td>Western US power grid</td>
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<td>9</td>
<td>6</td>
<td>9</td>
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<td>7</td>
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<td>4.40</td>
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<td>5</td>
<td>2</td>
<td>5</td>
<td>5.79</td>
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<td>2</td>
<td>3</td>
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<tr>
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<td>Little Rock food web</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1.59</td>
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</tbody>
</table>

\( L_c \) measures **local & absolute** (tree/clique) convexity in networks
convexity in graphs & networks

global convexity
tree/clique-like networks

regional convexity
core-periphery networks etc.

local convexity
random graphs
$< \ln n \ln \langle k \rangle$

c-convexity $\neq$ standard measures & c-core $\neq$ k-core
robustness, navigation, optimization, abstraction, comparison etc.
convex skeletons of networks

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convexity under randomization

\[ X_s = s - \sum_{t=1}^{sn-1} c \sqrt{\max(s \Delta s(t) - 1/n, 0)} \]

\[ s = \text{fraction of nodes in LCC} \]

\( X_s \) under **degree-preserving/full randomization** by edge rewiring

\( X_s \) very **sensitive** to **random perturbations** of network structure
convex skeletons of networks

convex skeleton = largest high-$X$s subnetwork (every $S$ is convex)

spanning tree & convex skeleton of network scientists coauthorships

convex skeleton is tree of cliques extracted by targeted edge removal
statistics of convex skeletons

\[ \langle C \rangle = \frac{1}{n} \sum_{i} \frac{2t_i}{k_i(k_i - 1)} \]

\[ \langle \sigma \rangle = \frac{2}{n(n-1)} \sum_{i<j} \sigma_{ij} \]

\[ X_s = \ldots \]

statistics of **convex skeletons** & **spanning trees** of networks

<table>
<thead>
<tr>
<th></th>
<th>clustering ( \langle C \rangle )</th>
<th>geodesics ( \langle \sigma \rangle )</th>
<th>convexity ( X_s )</th>
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<tbody>
<tr>
<td></td>
<td>N</td>
<td>CS</td>
<td>ST</td>
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<tr>
<td>Jazz musicians</td>
<td>0.62</td>
<td>0.81</td>
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<td>Network scientists</td>
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<td>Computer scientists</td>
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<td>Plasmodium falciparum</td>
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<td>Saccharomyces cerevisiae</td>
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<td>0.00</td>
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<td>Caenorhabditis elegans</td>
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<td>0.12</td>
<td>0.00</td>
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<tr>
<td>AS (January 1, 1998)</td>
<td>0.18</td>
<td>0.21</td>
<td>0.00</td>
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<tr>
<td>AS (January 1, 1999)</td>
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<td>0.27</td>
<td>0.00</td>
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<td>AS (January 1, 2000)</td>
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<td>0.25</td>
<td>0.00</td>
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<tr>
<td>Little Rock Lake</td>
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<td>0.69</td>
<td>0.00</td>
</tr>
<tr>
<td>Florida Bay (wet)</td>
<td>0.33</td>
<td>0.79</td>
<td>0.00</td>
</tr>
<tr>
<td>Florida Bay (dry)</td>
<td>0.33</td>
<td>0.82</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**convex skeleton** is generalization of **spanning tree** retaining **clustering**
distributions of convex skeletons

convex skeletons retain distributions in contrast to spanning trees of networks

convex skeletons & spanning trees of networks

Network scientists

AS (January 1, 1999)

Caenorhabditis elegans

fraction of nodes $p_k$
node degree $k$

fraction of nodes $p_k$
node degree $k$

fraction of nodes $p_k$
node degree $k$

fraction of nodes $p_d$
node distance $d$

fraction of nodes $p_d$
node distance $d$

fraction of nodes $p_d$
node distance $d$

convex skeletons retain distributions in contrast to spanning trees
convex skeletons of coauthorships

convex skeleton $\sim$ network abstraction technique

**convex skeleton** of Slovenian **computer scientists** coauthorships

computer theory (●), information systems (■), intelligent systems (▲), programming technologies (○) & other (●)
network backbones of coauthorships

convex skeleton $\gg$ high-betweenness & high-salience backbones

properties of backbones of Slovenian computer scientists coauthorships

![Graphs showing weight of coauthorship ties, modularity of field classification, and fraction of inter-field ties for different backbones](image)

convex skeletons enhance properties in contrast to other backbones
convex skeletons of networks

spanning tree

tree w/o cliques

convex skeleton

tree w/ cliques

convex skeleton $\gg$ backbones & c-centrality $\neq$ centralities

abstraction, sampling, visualization, modeling, dynamics etc.
thank you!

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ˇSubelj (2017) Convex skeletons of complex networks, pp. 19

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